$\therefore \alpha\beta+\gamma\delta=2b/a$. Since $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=m$, we have $(\alpha\beta+\gamma\delta)+(\alpha+\beta)(\gamma+\delta)=m$, or $(\alpha\beta+\gamma\delta)+(\alpha+\beta)^2=m$, or $2b/a+\frac{1}{4}a^2=m$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Place $x^4+ax^3+(2b/a+a^2/4)x^2+bx+c=(x^2+kx+m)(x^2+kx+n)$ where k is the sum of two of the roots. Since the sum of two roots equals the sum of the other two, k must be the same in both factors. Equating like powers of x we get $2k=a$, $k^2+m+n=2b/a+a^2/4$, $k(m+n)=b$, $mn=c$.

$\therefore k=a/2$, $m+n=2b/a$, $mn=c$, etc. The roots are now easily found—

$$m=\frac{1}{a}[b\pm\sqrt{(b^2-a^2c)}],\quad n=\frac{1}{a}[\mp\sqrt{(b^2-a^2c)}].$$

$$\therefore x=-\tfrac{1}{2}[k\mp\sqrt{(k^2-4m)}]=-\tfrac{1}{2}\left(\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{4}{b}[b\pm\sqrt{(b^2-a^2c)}]}\right).$$

$$x=-\tfrac{1}{2}[k\mp\sqrt{(k^2-4n)}]=-\tfrac{1}{2}\left(\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{4}{a}[b\mp\sqrt{(b^2-a^2c)}]}\right).$$

Analogously solved by F. D. Posey.

III. Solution by A. H. HOLMES, Brunswick, Maine, and L. E. NEWCOMB, Los Gatos, California.

Transposing c and adding b^2/a^2 to both sides, the roots of the equation are easily found to be

$$x_1=-\tfrac{1}{2}\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{4}{a}[b+\sqrt{(b^2-4ac)}]}\right]$$

$$x_2=-\tfrac{1}{2}\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{4}{a}[b+\sqrt{(b^2-4ac)}]}\right]$$

$$x_3=-\tfrac{1}{2}\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{4}{a}[b-\sqrt{(b^2-4ac)}]}\right]$$

$$x_4=-\tfrac{1}{2}\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{4}{a}[b-\sqrt{(b^2-4ac)}]}\right]$$

Evidently, $x_1+x_2=x_3+x_4=-\frac{1}{2}a$.

Also solved by Elmer Schuyler.

204. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If α, β, γ be the roots of the cubic equation $x^3+qx+r=0$, prove that $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

I. Solution by PHILIP GRABER, Ph. B., M. S., Akron, Ohio.

Since $\alpha+\beta+\gamma=0$ we may write $\alpha=-2.s$, $\beta=s+t$, $\gamma=s-t$. Then

$\Sigma\alpha^2=6s^2+2t^2$..........(1), $\Sigma\alpha^3=-6s^3+6st^2$..........(2),

$\Sigma\alpha^4=18s^4+12s^2t^2+2t^4$..........(3), and $\Sigma\alpha^5=-30s^5+20s^3t^2+10st^4$..........(4).

(1) multiplied by (4) gives $\Sigma\alpha^2\Sigma\alpha^5=-180s^7+60s^5t^2+100s^3t^4+20st^6$.........(5).

(2) multiplied by (3) gives $\Sigma\alpha^3\Sigma\alpha^4=-108s^7+36s^5t^2+60s^3t^4+12st^6$..........(6).

The second member of (5) multiplied by 3 equals the second member of (6) multiplied by 5.

$\therefore 3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

II. Solution by ELMER SCHUYLER, Reading, Pa.

By Newton's Theorem $\dfrac{xf'(x)}{f(x)}=n+\dfrac{\Sigma\alpha}{x}+\dfrac{\Sigma\alpha^2}{x^2}+$..........

In the present case $f(x)\equiv x^3+qx+r$, $f'(x)=3x^2+q$, and

$$\frac{xf'(x)}{f(x)}=3+\frac{0}{x}-\frac{2q}{x^2}-\frac{3r}{x^3}+\frac{2q^2}{x^4}+\frac{5qr}{x^5}\text{..........}$$

Consequently $\Sigma\alpha^2=-2q$, $\Sigma\alpha^5=5qr$, $\Sigma\alpha^3=-3r$, $\Sigma\alpha^4=2q^2$, and $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$ $=-30q^2r$.

III. Solution by J. SCHEFFER, Hagerstown, Md.

Since $\alpha+\beta+\gamma=0$, $\alpha\beta+\alpha\gamma+\beta\gamma=q$, $\alpha\beta\gamma=-r$, we find by squaring the first and applying the second, $\Sigma\alpha^2+2q=0$, whence $\Sigma\alpha^2=-2q$. By adding the three identities $\alpha^3+q\alpha+r=0$, $\beta^3+q\beta+r=0$, $\gamma^3+q\gamma+r=0$, we get $\Sigma\alpha^3+q\Sigma\alpha+3r=0$, or since $\Sigma\alpha=0$, $\Sigma\alpha^3=-3r$.

By adding the three identities $\alpha^4+q\alpha^2+r\alpha=0$, $\beta^4+q\beta^2+r\beta=0$, $\gamma^4+q\gamma^2+r\gamma=0$, we get $\Sigma\alpha^4+q\Sigma\alpha^2+r\Sigma\alpha=0$, whence $\Sigma\alpha^4=2q^2$.

By adding $\alpha^5+q\alpha^3+r\alpha^2=0$, $\beta^5+q\beta^3+r\beta^2=0$, $\gamma^5+q\gamma^3+r\gamma^2=0$, we get $\Sigma\alpha^5+q\Sigma\alpha^3+r\Sigma\alpha^2=0$, whence $\Sigma\alpha^5=5qr$.

$\therefore 3\Sigma\alpha^2.\Sigma\alpha^5=-30q^2r$, $5\Sigma\alpha^3.\Sigma\alpha^4=-30q^2r$, which proves the proposition.

IV. Solution by F. D. POSEY, A. B., San Mateo, Cal.

In the following, since $\Sigma\alpha=0$ the terms containing $\Sigma\alpha$ vanish.

$\Sigma\alpha^2=(\Sigma\alpha)^2-2\Sigma\alpha\beta=-2q$.

$\Sigma\alpha^3=(\Sigma\alpha^2)(\Sigma\alpha)-\Sigma\alpha^2\beta=-(\Sigma\alpha\beta)(\Sigma\alpha)+3\alpha\beta\gamma=-3r$.

$\Sigma a^4=(\Sigma a^3)(\Sigma a)-\Sigma a^3\beta=-(\Sigma a^2)(\Sigma a\beta)+\Sigma a^2\beta\gamma=-(\Sigma a^2)(\Sigma a\beta)+(\Sigma a)a\beta\gamma$
$=-(\Sigma a^2)(\Sigma a\beta)=2q^2.$

$\Sigma a^5=(\Sigma a^4)(\Sigma a)-\Sigma a^4\beta=-(\Sigma a^3)(\Sigma a\beta)+\Sigma a^3\beta\gamma=-(\Sigma a^3)(\Sigma a\beta)+(\Sigma a^2)a\beta\gamma=5rq.$

$\therefore 3\Sigma a^2\Sigma a^5=5\Sigma a^3\Sigma a^4.$

*** Solved by G. B. M. Zerr and Elmer Schuyler by computing the values of Σa^2, Σa^3, Σa^4, Σa^5 by means of the symmetric functions of the roots.

Solved by L. E. Newcomb by actual determination of the roots a, β, γ, of the given equation.

Also solved by G. W. Greenwood.

GEOMETRY.

226. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles ABC, $A'B'C'$ are in plane perspective, and the corresponding sides BC, $B'C'$, ..., cut in P, Q, R, respectively. AA', ..., cut the line PQR in $P'Q'R'$, respectively. Show that (PP', QQ', RR') is an involution range.

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

$B(PP', QR)=B(P'P, Q'R')$ since they both equal the same range $(B'P, DC)$ [D being the intersection of $C'B'$ and AB].

Therefore (PP', QR) and $(P'P, Q'R')$ are equicross and (PP', QQ', RR') is an involution range by the theorem that if (AA', BC), $(A'A, B'C')$ are equicross the range (AA', BB', CC') will be an involution. [Lachlan's *Modern Pure Geometry*, page 272, Art. 426].

229. Proposed by F. D. POSEY, A.B., San Mateo, Cal., and G. W. GREENWOOD, M.A. (Oxon), Lebanon, Ill.

The solutions of problem 219 in the April number, "devise a simple geometric solution of the general quadratic equation," give the roots when they are *real*. Required a construction for the roots when they are *complex*.

Solution by F. D. POSEY, A. B., San Mateo, Cal.

Dr. L. E. Dickson reports a solution on page 93 of the April issue of the MONTHLY which holds when the roots are real.

When the circle on AB does not cut Ox the roots are complex. From the center of the circle AB let fall a perpendicular upon Ox cutting the circle at C and Ox at P. Produce this line to D making $CPD=q+1$. On CD as diameter describe a circle cutting Ox at M and N. Then the roots of the equation are $OP+PM\sqrt{(-1)}$ and $OP-PM\sqrt{(-1)}$.

Proof. CP computed in terms of p and q is found to be

$$\frac{q+1-\sqrt{[p^2+(q-1)^2]}}{2}.$$